CLUSTER ALGEBRAS AND WEIL-PETERSSON FORMS

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ABSTRACT. In the previous paper [GSV] we have discussed Poisson properties of cluster algebras of geometric type for the case of a nondegenerate matrix of transition exponents. In this paper we consider the case of a general matrix of transition exponents. Our leading idea is that a relevant geometric object in this case is a certain closed 2-form compatible with the cluster algebra structure. The main example is provided by Penner coordinates on the decorated Teichmüller space, in which case the above form coincides with the classical Weil-Petersson symplectic form.

1. INTRODUCTION

Cluster algebras are a class of commutative rings defined axiomatically in terms of a distinguished family of generators called cluster variables. The study of cluster algebras was started by S. Fomin and A. Zelevinsky in [FZ2] and then continued in [FZ3], [FZ4], [GSV]. The main motivation for cluster algebras came from the study of dual canonical bases, see [BZ], [Ze2] and the theory of double Bruhat cells in complex semisimple Lie groups, see [SSV1], [SSV2], [FZ1], [SSVZ], [Ze1].

This paper is a continuation of [GSV], where we have developed a Poisson formalism compatible with cluster algebra structures. Here we suggest to associate with a cluster algebra a compatible skew-symmetric 2-form. Under certain irreducibility assumptions this 2-form is unique up to a scalar factor. In a non-degenerate case it is a symplectic form dual to the unique compatible Poisson structure constructed in [GSV]. In the general case there exists a projection of the cluster manifold defined in [GSV] to a rational manifold of a smaller dimension equipped with special τ-coordinates and their cluster transformations. On the latter manifold one has a natural compatible Poisson structure and a symplectic form dual to each other. Applying this construction to the coordinate ring of the Teichmüller space we recover

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the famous Weil-Petersson symplectic form. The initial cluster in this case is formed by *Penner coordinates* [Pe] associated with an ideal triangulation of a punctured Riemann surface. These coordinates proved to be a crucial ingredient in the study of decorated Teichmüller spaces [Pe], [Fo] and their quantization [Ka]. Motivated by this important example, we call the compatible symplectic form on the rational manifold associated with any cluster algebra *the Weil-Petersson symplectic form*.

Since the earlier version of this paper was posted, two preprints by Fock and Goncharov [FG1], [FG2] has appeared that explored cluster algebra aspects of the moduli space of local systems on a punctured Riemann surface.

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2. Degenerate cluster algebras of geometric type

Let $\mathcal{A}$ be a cluster algebra of geometric type. As it has been discussed in [FZ2], $\mathcal{A}$ is completely determined by the initial cluster with a fixed set of cluster variables $x_1, \ldots, x_n$ and a skew-symmetrizable transformation matrix Z. In our previous paper [GSV] we have considered the case of nondegenerate Z. This paper discusses the case of general Z. For the sake of simplicity we consider only the skew-symmetric case.

We start by recalling main definitions from [FZ2] and [GSV].

2.1. Cluster algebras of meromorphic functions on a rational manifold. Let $\mathcal{Z}_n$ be the set of integer-valued skew-symmetric $n \times n$ matrices. According to [FZ2], any $Z = (z_{ij}) \in \mathcal{Z}_n$ defines a cluster algebra of geometric type in the following way. Let us fix a set of n *cluster* variables $f_1, \ldots, f_n$. For each $i \in [1, n]$ we introduce a transformation T_i of cluster variables by

$$(2.1) \qquad T_i(f_j) = \bar{f}_j = \begin{cases} \dfrac{1}{f_i} \left(\displaystyle\prod_{z_{ik} > 0} f_k^{z_{ik}} + \prod_{z_{ik} < 0} f_k^{-z_{ik}} \right) & \text{for } j = i \\ f_j & \text{for } j \neq i, \end{cases}$$

and the corresponding matrix transformation $\bar{Z} = T_i(Z)$, called *mutation*, by

$$(2.2) \qquad \bar{z}_{kl} = \begin{cases} -z_{kl} & \text{for } (k-i)(l-i) = 0 \\ z_{kl} + \dfrac{|z_{ki}|z_{il} + z_{ki}|z_{il}|}{2} & \text{for } (k-i)(l-i) \neq 0. \end{cases}$$

Observe that T_i is an involution and $\bar{Z}$ belongs to $\mathcal{Z}_n$. Thus, one can apply transformations T_i to the new set of cluster variables (using the new matrix), etc. The *rank n cluster algebra* (*of geometric type*) is the subalgebra of the field of rational functions in cluster variables $f_1, \ldots, f_n$ generated by the union of all clusters; its ground ring is the ring of integers. We denote this algebra by $\mathcal{A}(Z)$.

One can represent $\mathcal{A}(Z)$ with the help of an n-regular tree $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \ldots, n$ so that the n edges incident to each vertex receive

different labels. To each vertex v of $\mathbb{T}_n$ we assign a set of n variables $f_{v,1}, \ldots, f_{v,n}$, which together form a *cluster* at v. For an edge $v\bar{v}$ of $\mathbb{T}_n$ that is labeled by $i \in [1, n]$, the clusters $f = f_v$ and $\bar{f} = f_{\bar{v}}$ are related by the transformation T_i given by (2.1).

Example 1. Consider a 3×3 skew-symmetric matrix

$$
Z = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}
$$

and fix the initial cluster f_1, f_2, f_3. Then the action of transformations T_1, T_2, T_3 is given by

$$
T_1(f_1) = \frac{f_2 + f_3}{f_1}, \quad T_1(f_2) = f_2, \quad T_1(f_3) = f_3, \qquad T_1(Z) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix};
$$

$$
T_2(f_2) = \frac{f_1 + f_3}{f_2}, \quad T_2(f_1) = f_1, \quad T_2(f_3) = f_3, \qquad T_2(Z) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix};
$$

$$
T_3(f_3) = \frac{f_1 + f_2}{f_3}, \quad T_3(f_1) = f_1; \quad T_3(f_2) = f_2, \qquad T_3(Z) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.
$$

Consider an n-dimensional rational manifold $\mathcal{M}$ (recall that this means that the field $F(\mathcal{M})$ of meromorphic functions on $\mathcal{M}$ is a transcendental extension of the ground field, i.e. the field of rational functions in n variables). Let us fix an arbitrary set of n generators $\{x_1, \ldots, x_n\}$ of the field $F(\mathcal{M})$. Then we get an embedding of the cluster algebra $\mathcal{A}(Z)$ into $F(\mathcal{M})$ by mapping the initial cluster variables $f_1, \ldots, f_n$ to $x_1, \ldots, x_n$, respectively. In what follows we assume that this embedding is fixed and identify the elements of $\mathcal{A}(Z)$ with the corresponding meromorphic functions on $\mathcal{M}$.

Recall that in our previous paper [GSV] we have considered a slightly different situation: only a part of the variables is regarded as cluster variables; the transformations of the remaining variables are prohibited, and these variables are called *stable*. In this situation only m rows of the transformation matrix are given, where m is the number of cluster variables. An important nondegeneracy condition assumed in [GSV] is that the rank of the corresponding $m \times n$ matrix equals m (it is proved in [GSV] that the rank is invariant under mutations).

In our present situation, when Z is an $n \times n$ matrix of rank $m \leqslant n$, one can choose m rows of Z so that the resulting submatrix of Z is nondegenerate, and obtain the situation described in [GSV]. The corresponding cluster algebra is then a subalgebra of $\mathcal{A}(Z)$; it is denoted $\mathcal{A}_I(Z)$, where $I = (i_1, \ldots, i_m)$ is the multiindex corresponding to the chosen rows.

On the other hand, starting from a cluster algebra with both cluster and stable variables, one can get a coefficient-free cluster algebra (the one without stable variables) by just fixing the values of all the stable variables at 1.

2.2. Compatible Poisson brackets. Let $\{\cdot, \cdot\}$ be a Poisson bracket on an n-dimensional algebraic manifold $\mathcal{M}$. We say that functions $g_1, \ldots, g_n$ are *log-canonical* with respect to $\{\cdot, \cdot\}$ if $\{g_i, g_j\} = p_{ij} g_i g_j$, where p_{ij} are constants. The

matrix (p_{ij}) is called the *coefficient matrix* of $\{\cdot,\cdot\}$ (in the basis g); evidently, it is skew-symmetric. Assume that $\mathcal{M}$ is a rational manifold. We say that a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{M}$ is *compatible* with a cluster algebra $\mathcal{A}(Z)$ if all clusters in $\mathcal{A}(Z)$ are log-canonical with respect to $\{\cdot,\cdot\}$.

In [GSV] we studied the space of Poisson brackets compatible with algebras of type $\mathcal{A}_I(Z)$. In particular, we have proved that such brackets (considered up to a scalar factor) form a vector space of dimension $1 + \binom{n-m}{2}$, where m is the rank of Z. As a consequence, when Z is nondegenerate, there exists a unique (up to a scalar factor) Poisson bracket compatible with $\mathcal{A}(Z)$.

Examples show that when a cluster algebra is associated with a geometric object, the corresponding compatible Poisson brackets also have a natural invariant interpretation. This is the case, e.g., for cluster algebras associated with double Bruhat cells in a semi-simple Lie group. There, the initial cluster can be chosen to consist of certain generalized minors. These minors form a log-canonical coordinate system with respect to the standard Poisson-Lie structure on the group. Moreover, the corresponding Poisson bracket is compatible with cluster transformations (see [FZ1], [KZ] for definitions and details). Another example, discussed in [GSV], is the real Grassmannian $G(k,n)$, viewed as a Poisson homogeneous space of the group $SL(n)$ equipped with the Sklyanin Poisson-Lie bracket. A log-canonical coordinate system for a push-forward of the Sklyanin bracket to $G(k,n)$ is formed by a subset of Plücker coordinates. These coordinates serve as an initial cluster for the cluster algebra compatible with this Poisson bracket. It is proved in [GSV] that the resulting cluster algebra contains all the Plücker coordinates and thus, the coordinate ring of $G(k,n)$ is endowed with a cluster algebra structure.

However, the following example shows that a bracket compatible with $\mathcal{A}(Z)$ may not exist if Z is degenerate.

Example 2. Let $\{\cdot,\cdot\}$ be a Poisson structure compatible with the cluster algebra described above in Example 1. Then, by the definition of compatibility, $\{f_1, f_2\} = \lambda f_1 f_2$, $\{f_2, f_3\} = \mu f_2 f_3$, $\{f_1, f_3\} = \nu f_1 f_3$. Applying T_1 one gets $\{\bar{f}_1, f_2\} = -\lambda f_2^2/f_1 - \mu f_2 f_3/f_1 - \lambda f_2 f_3/f_1$. On the other hand, the compatibility yields $\{\bar{f}_1, f_2\} = \alpha \bar{f}_1 f_2 = \alpha f_2^2/f_1 + \alpha f_2 f_3/f_1$. Comparing these two expressions we immediately get $\mu = 0$, which means that $\{f_2, f_3\} = 0$. Similarly, $\{f_1, f_2\} = \{f_1, f_3\} = 0$. Hence we see that the only Poisson structure compatible with this cluster algebra is trivial.

2.3. Compatible 2-forms. Motivated by the example above, we will look for an alternative geometric object compatible with a general cluster algebra. Natural duality suggests a compatible differential 2-form as a possible candidate.

Let ω be a differential 2-form on an n-dimensional algebraic manifold. We say that functions $g_1, \ldots, g_n$ are *log-canonical* with respect to ω if

$$\omega = \sum_{i,j=1}^n \omega_{ij} \frac{dg_i \wedge dg_j}{g_i g_j},$$

where ω_{ij} are constants. The matrix $\Omega = (\omega_{ij})$ is called the *coefficient matrix* of ω (in the basis g); evidently, Ω is skew-symmetric. We say that a 2-form ω on a rational manifold is *compatible* with a cluster algebra $\mathcal{A}(Z)$ if all clusters in $\mathcal{A}(Z)$ are log-canonical with respect to ω.

We say that a skew-symmetric matrix M is *reducible* if there exists a permutation matrix P such that PMP^T is a block-diagonal matrix, and *irreducible* otherwise; $r(M)$ is defined as the maximal number of diagonal blocks in PMP^T. The partition into blocks defines an obvious equivalence relation $\sim$ on the rows (or columns) of M.

Theorem 2.1. *The 2-forms on a rational n-dimensional manifold compatible with $\mathcal{A}(Z)$ form a vector space of dimension $r(Z)$. Moreover, the coefficient matrices of these 2-forms in the basis formed by the cluster variables are characterized by the equation $\Omega = \Lambda Z$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = \lambda_j$ whenever $i \sim j$. In particular, if Z is irreducible, then $\Omega = \lambda Z$.*

Proof. Indeed, let ω be a 2-form compatible with $\mathcal{A}(Z)$. Then

$$\omega = \sum_{j,k=1}^{n} \omega_{jk} \frac{df_j \wedge df_k}{f_j f_k} = \sum_{j,k=1}^{n} \bar{\omega}_{jk} \frac{d\bar{f}_j \wedge d\bar{f}_k}{\bar{f}_j \bar{f}_k},$$

where $\bar{f}_j = T_i(f_j)$ is given by (2.1) and $\bar{\omega}_{jk}$ are the coefficients of ω in the basis $\bar{f}$. Recall that the only variable in the cluster $\bar{f}$ that is different from the corresponding variable in f is $\bar{f}_i$, and

$$d\bar{f}_i = -\frac{1}{f_i^2} \left(\prod_{z_{ik}>0} f_k^{z_{ik}} + \prod_{z_{ik}<0} f_k^{-z_{ik}} \right) df_i +$$
$$\sum_{z_{ik}>0} \frac{z_{ik}}{f_k} \frac{1}{1 + \prod_{k=1}^{n} f_k^{-z_{ik}}} df_k - \sum_{z_{ik}<0} \frac{z_{ik}}{f_k} \frac{1}{1 + \prod_{k=1}^{n} f_k^{z_{ik}}} df_k.$$

Thus, for any $j \neq i$ we immediately get

(2.3a) $$\bar{\omega}_{ij} = -\omega_{ij}.$$

Next, consider any pair $j, k \neq i$ and assume that both z_{ij} and z_{ik} are positive. Then

$$\omega_{jk} = \bar{\omega}_{jk} + \frac{\bar{\omega}_{ik} z_{ij} + \bar{\omega}_{ji} z_{ik}}{1 + \prod_{k=1}^{n} f_k^{-z_{ik}}}.$$

This equality can only hold if $\bar{\omega}_{ik} z_{ij} + \bar{\omega}_{ji} z_{ik} = 0$, which gives

(2.4) $$\frac{\omega_{ij}}{z_{ij}} = \frac{\omega_{ik}}{z_{ik}} = \lambda_i.$$

Besides, in this situation

(2.3b) $$\bar{\omega}_{jk} = \omega_{jk}.$$

Similarly, if both z_{ij} and z_{ik} are negative, then

$$\omega_{jk} = \bar{\omega}_{jk} - \frac{\bar{\omega}_{ik} + \bar{\omega}_{ji} z_{ik}}{1 + \prod_{k=1}^{n} f_k^{z_{ik}}},$$

and hence the same two relations as above are true.

Finally, let z_{ij} and z_{ik} have different signs, say, $z_{ij} > 0$ and $z_{ik} < 0$, then

$$\omega_{jk} = \bar{\omega}_{jk} + \frac{\bar{\omega}_{ik} z_{ij} \prod_{k=1}^n f_k^{z_{ik}} - \bar{\omega}_{ji} z_{ik}}{1 + \prod_{k=1}^n f_k^{z_{ik}}},$$

which again leads to (2.4); the only difference is that in this case

$$(2.3c) \qquad \bar{\omega}_{jk} = \omega_{jk} + \omega_{ik} z_{ij}.$$

By (2.4), $\Omega = \Lambda Z$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Since both Ω and Z are skew-symmetric, we immediately get $\lambda_i = \lambda_j$ whenever $i \sim j$.

It is worth mentioning that relations (2.3a-c) are equivalent to (2.2). $\square$

2.4. τ-coordinates and the Weil-Petersson form. Assume now that $\mathcal{M} = \mathcal{X}$ is the cluster manifold defined in [GSV]. Recall that for a rank n cluster algebra over a field $k = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{X}$ is obtained by gluing together toric charts $(k^*)^n$ corresponding to clusters. The corresponding cluster variables give the distinguished coordinate system in each chart; the gluing is defined by (2.1). By [GSV, Lemma 2.1], $\mathcal{X}$ is rational. Let ω be one of the forms described in Theorem 2.1. Note that ω is degenerate whenever $m < n$. Kernels of ω form an integrable distribution (since it becomes linear in the coordinates $\log f$), so one can define the quotient mapping $\pi^\omega \colon \mathcal{X} \to \mathcal{X}^\omega$, where $\mathcal{X}^\omega$ is a rational manifold of dimension m. Observe that this mapping in fact does not depend on the choice of ω, provided ω is generic (that is, the corresponding diagonal matrix Λ is nonsingular), since the kernels of all generic ω coincide. The symplectic reduction $\pi_*^\omega \omega$ of ω is thus well defined; moreover, $\pi_*^\omega \omega$ is a symplectic form on the image $\mathcal{X}^\omega$.

Let us introduce coordinates in the image that are naturally related to the initial coordinates f on $\mathcal{X}$. To do this, we define another n-tuple of rational functions on $\mathcal{X}$; it is denoted $\tau = (\tau_1, \dots, \tau_n)$ and is given by

$$\tau_j = \prod_{k=1}^n f_k^{z_{jk}}, \qquad j \in [1, n].$$

Given a cluster f, we say that the entries of the corresponding τ form a τ-cluster. It is easy to see that elements of the τ-cluster are no longer functionally independent. To get a functionally independent subset one has to choose at most m entries in such a way that the corresponding rows of Z are linearly independent. Transformations of τ-coordinates corresponding to cluster transformations (2.1) are studied in [GSV].

Clearly, any choice of m functionally independent coordinates τ provides log-canonical coordinates for the 2-form $\pi_*^\omega \omega$. Besides, $\pi_*^\omega \omega$ is compatible with the cluster algebra transformations. Moreover, if γ is any other 2-form on the image $\mathcal{X}^\omega$ compatible with the cluster algebra transformations, then its pullback is a compatible 2-form on the preimage $\mathcal{X}$. Thus, we get the following result.

Corollary 2.2. *Let Z be irreducible, then there exists a unique, up to a constant factor, 2-form on $\mathcal{X}^\omega$ compatible with $\mathcal{A}(Z)$. Moreover, this form is symplectic.*

The unique symplectic form described in the above corollary is called the *Weil-Petersson* form associated with the cluster algebra $\mathcal{A}(Z)$. The reasons for this name are explained in the Introduction.

In our previous paper [GSV] we have studied Poisson structures compatible with a cluster algebra. Fix a multiindex I and consider the reduced cluster algebra $\mathcal{A}_I(Z)$, as explained at the end of Section 2.1. By Theorem 1.4 of [GSV], there exists a linear space of Poisson structures compatible with $\mathcal{A}_I(Z)$. To distinguish a unique, up to a constant factor, Poisson structure, we consider the restriction of the form ω to the affine subspace $f_i = \text{const}$, $i \notin I$. This restriction is a symplectic form, since the kernels of ω are transversal to the above subspace, and it is compatible with $\mathcal{A}_I(Z) \subset \mathcal{A}(Z)$. Its dual is therefore a Poisson structure on this affine subspace, compatible with $\mathcal{A}_I(Z)$. By the same transversality argument, we can identify each affine subspace $f_i = \text{const}$, $i \notin I$, with the image of the quotient map π_Z, and hence each of the obtained Poisson structures can be identified with the dual to the Weil-Petersson symplectic form.

In [GSV] we have considered an alternative description of the Poisson structure dual to $\pi_*^\omega \omega$. It follows from the proof of Theorem 1.4 in [GSV] that it is the unique (up to a nonzero scalar factor) Poisson structure P on $\mathcal{X}^\omega$ satisfying the following condition: for any Poisson structure on $\mathcal{X}$ compatible with any reduced cluster algebra $\mathcal{A}_I(Z)$, the projection $p \colon \mathcal{X} \to \mathcal{X}^\omega$ is Poisson. The basis τ_i is log-canonical with respect to P, and $\{\tau_i, \tau_j\} = \lambda z_{ij} \tau_i \tau_j$ for some constant λ.

3. Main example: Teichmüller space

Let Σ be a Riemann surface of genus g with s punctures (marked points).

3.1. Teichmüller space, horocycles, decorated Teichmüller space. Let $\mathcal{T}_g^s$ denote the Teichmüller space of marked complete metrics on Σ having constant negative curvature -1 and a finite area, modulo the action of the connected component of identity in the group of diffeomorphisms of Σ. By the uniformization theorem, Σ is identified with the quotient of the complex upper half-plane modulo the action of a discrete Möbius group. A horocycle centered at a point p at the absolute is a circle orthogonal to any geodesic passing through p. All horocycles centered at some fixed point p can be parametrized by a positive real number called the *height* of the horocycle. The *decorated* Teichmüller space $\widetilde{\mathcal{T}}_g^s$ classifies hyperbolic metrics on Σ with a chosen horocycle around each of the marked points; $\widetilde{\mathcal{T}}_g^s$ is a fiber bundle over $\mathcal{T}_g^s$ whose fiber is $\mathbb{R}_+^s$.

More exactly, let V be a Minkowsky 3-space, i.e. a real vector space of dimension 3 with a nondegenerate quadratic form $\langle \cdot, \cdot \rangle$ of type $(2,1)$. Choosing appropriate coordinates, one can write the Minkowsky metric as $-dv_0^2 + dv_1^2 + dv_2^2$. Consider the two-sheeted hyperboloid $\{v \in V \colon \langle v, v \rangle = -1\}$, and denote by $\mathbb{H}$ its upper sheet. It is well known that $\mathbb{H}$ is isometric to the Poincaré disk model for the hyperbolic plane. An explicit isometry is given by the radial projection from the origin to the unit disk D about $(1,0,0)$. For $x, y \in \mathbb{H}$, the Poincaré distance d between the projections of x and y to D satisfies $\cosh^2 d = \langle x, y \rangle^2$.

The light cone L is defined by $L = \{v \in V \colon \langle v, v \rangle = 0\} = \{v \in V \colon v_0^2 = v_1^2 + v_2^2\}$. The positive light cone L^+ is given by $L^+ = \{v \in L \colon v_0 > 0\}$. The radial projection extends to a map $\pi \colon \mathbb{H} \cup L^+ \to D \cup S_\infty^1$, where S_∞^1 is the boundary circle of D. A point $w \in L^+$ corresponds to the horocycle $\{x \in \mathbb{H} \colon \langle w, x \rangle = -1\}$ of height w_0 about $\pi(w)$.

Let us recall here a construction of coordinates on the decorated Teichmüller space (see [Pe1]).

Fix any ideal geodesic triangulation Δ with vertices at the marked points. Recall that an ideal geodesic triangulation is a maximal family of disjointly embedded simple geodesic arcs subject to the condition that no complementary region of Δ in Σ is a monogon or a bigon (see Fig. 1). An ideal triangulation subdivides Σ into open pieces (triangles) which are images of ideal triangles in the complex upper-half plane under the uniformization map. Observe that Δ may contain triangles with two coinciding sides (see Fig. 1). Any edge $e \in \text{Edge}(\Delta)$ of this triangulation has an infinite hyperbolic length, however the hyperbolic length of its segment between the corresponding horocycles is finite. Let us denote by $l(e)$ the signed length of this segment, i.e., we take the length of e with the positive sign if the horocycles do not intersect, and with the negative sign if they do, and define $f(e) = \exp(l(e)/2)$. Equivalently, a choice of a geodesic between two marked points corresponding to an edge e together with two horocycles determines two points x, y on the positive light cone, and we put $f(e) = \sqrt{-\langle x, y \rangle}$.
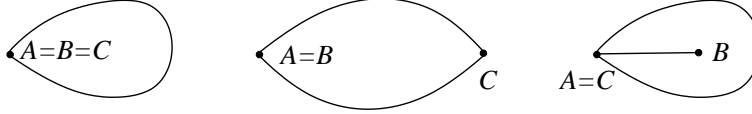

FIG. 1. MONOGON, BIGON, AND A TRIANGLE WITH TWO COINCIDING SIDES

The following result is well known (see [Pe1]).

Theorem 3.1. *For any ideal triangulation Δ, the functions $f(e)$, $e \in \text{Edge}(\Delta)$, define a homeomorphism between the decorated Teichmüller space $\widetilde{\mathcal{T}}_g^s$ and $\mathbb{R}_+^{6g-6+3s}$.*

The coordinates described in Theorem 3.1 are called the *Penner coordinates* on $\widetilde{\mathcal{T}}_g^s$.

3.2. Whitehead moves and the cluster algebra structure. Given a triangulation of Σ, one can obtain a new triangulation via a sequence of simple transformations called flips, or Whitehead moves, see Fig. 2. Note that some of the sides a, b, c, d can coincide, while p is distinct from any of them.
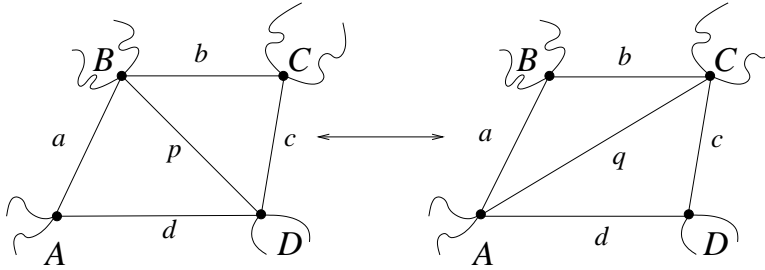

FIG. 2. WHITEHEAD MOVE

Under such a transformation the Penner coordinates undergo the following change described by the famous *Ptolemy relation*:

$$(3.1) \qquad\qquad f(p)f(q) = f(a)f(c) + f(b)f(d).$$

Note that this transformation looks very much like a cluster algebra transformation; we would like to make this statement more precise. Let us call an ideal triangulation of Σ *nice* if all the edges of any triangle are pairwise distinct. We call a nice triangulation *perfect* if, moreover, all vertices have at least three incident half-edges (in other words, loops are counted with multiplicity 2). We will consider Penner coordinate systems only for nice triangulations. Note that every edge in a perfect triangulation borders two distinct triangles with different edges. So, a and c can coincide with each other, but not with any of b and d; similarly, b and d can coincide as well.

Therefore in this situation the Ptolemy relation holds for any edge. In general, flips do not preserve the perfectness of a triangulation; however, the result of a flip of a perfect triangulation is a nice one.

Given a perfect triangulation Δ, we construct the following coefficient-free cluster algebra $\mathcal{A}(\Delta)$: the Penner coordinates are cluster coordinates, and the transformation rules are defined by Ptolemy relations. The transformation matrix $Z(\Delta)$ is determined by Δ in the following way.

Let $\nu_P(a, b)$ be the number of occurrences of the edge b immediately after a in the counterclockwise order around vertex P. For any pair of edges $a, b \in \mathrm{Edge}(\Delta)$, put

$$Z(\Delta)_{ab} = \sum_{P \in \mathrm{Vert}(\Delta)} (\nu_P(a, b) - \nu_P(b, a)),$$

and define $\mathcal{A}(\Delta) = \mathcal{A}(Z(\Delta))$.

For example, the left part of Fig. 3 shows a triangulation of the sphere with three marked points. Here $\nu_P(a, b) = \nu_P(b, a) = 1$ and $\nu_Q(a, b) = \nu_Q(b, a) = \nu_R(a, b) = \nu_R(b, a) = 0$, hence $Z(\delta)_{ab} = 0$. The right part of Fig. 3 shows a triangulation of the torus with one marked point. Here $\nu_P(a, b) = 0$, $\nu_P(b, a) = 2$, and hence $Z(\Delta)_{ab} = -2$.
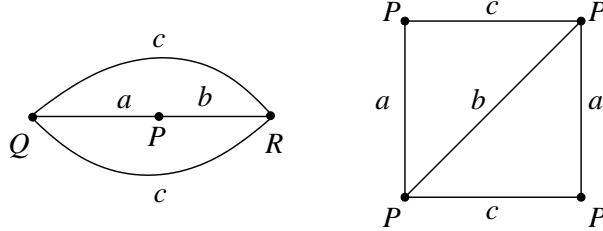

FIG. 3. TO THE DEFINITION OF $Z(\Delta)$

Let $\Delta \mapsto \bar{\Delta}$ be a flip shown on Fig. 2. By the definition of $\mathcal{A}(\Delta)$ we see that cluster variables in $\mathcal{A}(\Delta)$ corresponding to the cluster obtained from the initial one by the transformation $T \colon f(p) \mapsto f(q)$ coincide with Penner coordinates for the flipped triangulation $\bar{\Delta}$. To check that the flip is indeed a cluster algebra transformation, it is enough to check that the transformation matrix $\bar{Z}$ is determined by the adjacency of edges in $\bar{\Delta}$ in the same way as $Z = Z(\Delta)$ above.

To prove this, recall that the new transformation matrix $\bar{Z}$ is obtained from Z by the mutation rule (2.2). Assume first that the degrees of both endpoints of p are at least three; the Whitehead move in this case is shown on Fig. 2.

Note that the image of the diagonal p under the flip is the diagonal q. Using formula (2.2) we obtain $\bar{z}_{ad} = z_{ad} - 1$, $\bar{z}_{ba} = z_{ba} + 1$, $\bar{z}_{cb} = z_{cb} - 1$, $\bar{z}_{dc} = z_{dc} + 1$, $\bar{z}_{xq} = -z_{xp}$, where x runs over $\{a, b, c, d\}$.

For instance, substituting the values $z_{da} = z_{ap} = z_{pd} = z_{cp} = z_{pb} = z_{bc} = 1$, $z_{ab} = z_{cd} = 0$ into (2.2) we obtain $\bar{z}_{ab} = \bar{z}_{bq} = \bar{z}_{qa} = \bar{z}_{dq} = \bar{z}_{qc} = \bar{z}_{cd} = 1$ and $\bar{z}_{ad} = \bar{z}_{bc} = 0$ which corresponds to the adjacency rules of Fig. 2. Similarly, one can easily check that the adjacency rules hold for other triangulations containing the rectangle $ABCD$.

This shows that Penner coordinates for two perfect triangulations related by a Whitehead move form two sets of cluster coordinates for adjacent clusters in $\mathcal{A}(\Delta)$.

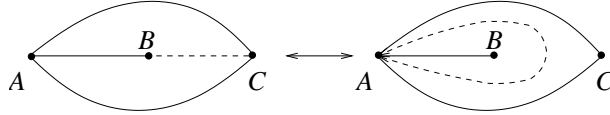A problem arises in the case of non-perfect triangulations, as shown in Fig. 4.

FIG. 4. PROHIBITED WHITEHEAD MOVE

Indeed, Ptolemy relation implies $f_{BC}f_{A\bigcirc A} = f_{AB}(f_{A\frown C} + f_{A\smile C})$ However, by the definition of cluster algebras, the right part of the cluster algebra relation must be irreducible, and in fact the cluster algebra relation are $f_{BC}f_{A\bigcirc A} = f_{A\frown C} + f_{A\smile C}$. We thus see that the flip with respect to the edge BC breaks the cluster algebra rules, meaning that Penner coordinates after the flip differ from the coordinates prescribed by the cluster algebra rules.

To overcome this problem we are going to prohibit flips of certain diagonals in non-perfect triangulations. Namely, we call a Whitehead move *allowed* only if both endpoints of the flipped diagonal before the move have degrees at least three. As we have already mentioned, the transformation rule for Penner coordinates coincides with the transformation rule for the corresponding cluster algebra, and we are guaranteed that Penner coordinates of triangulations obtained by a sequence of such flips form clusters in the same cluster algebra $\mathcal{A}(\Delta)$.

Observe that all nice triangulations might be obtained one from another by a sequence of allowed flips. This can be proved by an immediate modification of the proofs provided in [Ha, Bu], where the authors consider the same property for ideal triangulations [Ha], or all triangulations [Bu]; another proof follows from the fact that nice triangulations label maximal simplices of triangulations of the Teichmüller space, see [Iv] and references therein. Therefore, cluster algebras $\mathcal{A}(\Delta)$ defined by all perfect triangulations Δ of a fixed punctured surface are isomorphic to the same cluster algebra, which we denote by $\mathcal{A}(\Sigma)$.

The clusters of $\mathcal{A}(\Sigma)$ corresponding to nice triangulations are called *Teichmüller clusters*. Let $\mathcal{X}(\Sigma)$ be the cluster manifold for the cluster algebra $\mathcal{A}(\Sigma)$. It is easy to see that the decorated Tecimüller space $\tilde{\mathcal{T}}_g^s$ coincides with the positive part of any toric chart defined by a Teichmüller cluster.

3.3. The Weil-Petersson form. Recall that if Δ is a perfect triangulation, then flips of all edges are allowed. Let us define the *star* of Δ as the subset of clusters in $\mathcal{A}(\Sigma)$ formed by Δ itself and all the clusters obtained by flips with respect to all edges of Δ; we denote the star of Δ by $\star(\Delta)$. It follows immediately from the proof

of Theorem 2.1 and the connectedness of the graph of Δ that there exists a unique (up to a constant) differential 2-form on $\mathcal{X}(\Sigma)$ compatible with all clusters in $\star(\Delta)$; moreover, this form is compatible with the whole algebra $\mathcal{A}(\Sigma)$. Recall that this form is given in coordinates $f(e)$, $e \in \mathrm{Edge}(\Delta)$, by the following expression:

$$(3.2) \qquad \omega = \mathrm{const} \cdot \sum_{b \curvearrowleft a} \frac{df(a) \wedge df(b)}{f(a)f(b)},$$

where $b \curvearrowleft a$ means that edge b follows immediately after a in the counterclockwise order around some vertex of Δ.

It is well-known that the Teichmüller space $\mathcal{T}_g^s$ is a symplectic manifold with respect to the famous Weil-Petersson symplectic form W. Following [Pe2], we recall here the definition of W. The cotangent space $T_\Sigma^* \mathcal{T}_g^s$ is formed by holomorphic quadratic differentials $\phi(z)dz^2$. Define the Weil-Petersson nondegenerate co-metric by the following formula:

$$\langle \phi_1(z)dz^2, \phi_2(z)dz^2 \rangle = \frac{i}{2} \int_\Sigma \frac{\phi_1(z)\phi_2(z)}{\lambda(z)} dz \wedge \overline{dz},$$

where $\lambda(z)|dz^2|$ is the hyperbolic metric on Σ. The Weil-Petersson metric is defined by duality. Moreover, the Weil-Petersson metric is Kähler. Hence, its imaginary part is a symplectic 2-form W, which is called the Weil-Petersson symplectic form.

As we have already mentioned above, the decorated Teichmüller space is fibered over $\mathcal{T}_g^s$ with a trivial fiber $\mathbb{R}_{>0}^s$. The projection $\rho \colon \tilde{\mathcal{T}}_g^s \to \mathcal{T}_g^s$ is given by forgetting the horocycles.

Given a geodesic triangulation, we introduce *Thurston shear coordinates* on the Teichmüller space. Following [Th, Pe1, Fo], we associate to each edge of the triangulation a real number. Choose an edge e and two triangles incident to it and lift the resulting rectangle to the upper half plane. Vertices of this geodesic quadrilateral lie on the real axis. We obtain therefore four points on the real axis, or more precisely, on $\mathbb{R}P^1$. Among the constructed four points there are two distinguished ones, which are the endpoints of the edge e we have started with. Using the Möbius group action on the upper half plane, we can shift these points to zero and infinity, respectively, and one of the remaining points to -1. Finally, we assign to e the logarithm of the coordinate of the fourth point (the fourth coordinate itself is a suitable cross-ratio of those four points).

To show that we indeed obtained coordinates on the Teichmüller space, it is enough to reconstruct the surface. We will glue the surface out of ideal hyperbolic triangles. The lengths of the sides of ideal triangles are infinite, and therefore we can glue two triangles in many ways which differ by shifting one triangle w.r.t. another one along the side. The ways of gluing triangles can be parametrized by the cross-ratio of four vertices of the obtained quadrilateral (considered as points of $\mathbb{R}P^1$).

By [Pe1, Fo], the projection ρ is written in these natural coordinates as follows:

$$g(e) = \log f(e_1) + \log f(e_3) - \log f(e_2) - \log f(e_4),$$

where $\{g(e) \colon e \in \mathrm{Edge}(\Delta)\}$ and $\{f(e) \colon e \in \mathrm{Edge}(\Delta)\}$ are Thurston shear coordinates on the Teichmüller space and Penner coordinates on the decorated Teichmüller space, respectively, with respect to the same triangulation Δ; notation e_i is explained in Fig. 5.
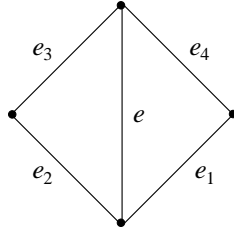
FIG. 5. NOTATION FOR NEIGHBORING EDGES

We see that $\exp(\rho)$ coincides with the transition from cluster variables to τ-coordinates in the cluster algebra $\mathcal{A}(\Sigma)$. Therefore, the pullback of the Weil-Petersson symplectic form determines a degenerate 2-form of corank s on the decorated Teichmüller space. For any nice triangulation Δ its expression in terms of Penner coordinates $f(e)$, $e \in \mathrm{Edge}(\Delta)$, is as follows:

$$\omega = \frac{1}{2} \sum_{e \in \mathrm{Edge}(\Delta)} (dx_{e_1} \wedge dx_e - dx_{e_2} \wedge dx_e + dx_{e_3} \wedge dx_e - dx_{e_4} \wedge dx_e),$$

where $x_e = \log f(e)$, see for instance [FR].

Comparing this expression with (3.2) we can summarize the previous discussion as follows.

Theorem 3.2. (i) *The decorated Teichmüller space $\tilde{\mathcal{T}}_g^s$ is the subset of the cluster manifold $\mathcal{X}(\Sigma)$ given by the positive part of the toric chart defined by any Teichmüller cluster.*

(ii) *The projection to the τ-coordinates is the forgetful map from the decorated Teichmüller space to the Teichmüller space.*

(iii) *The corank of the corresponding transformation matrix equals s.*

(iv) *The unique compatible symplectic structure induced on the Teichmüller space is the classical Weil-Petersson symplectic structure.*

In the next section, using the cluster algebra approach, we describe subsystems of Penner coordinates such that restriction of ω to these subsystems is nondegenerate. In particular, we obtain an independent combinatorial proof of Theorem 3.2(iii). Choosing complementary coordinates as stable variables and following the procedure suggested in [GSV], one obtains the Weil-Petersson Poisson structure on the decorated Teichmüller space.

3.4. Gauge group action. Given a triangulation Δ of Σ, consider the transformation matrix $Z(\Delta)$ for the cluster algebra $\mathcal{A}(\Delta)$. Our goal is to prove that the corank of $Z(\Delta)$ equals s and to choose a subset of Penner coordinates such that the restriction of Z onto these coordinates is nondegenerate.

Let us prove first that $\operatorname{corank} Z(\Delta) \geqslant s$. Indeed, the gauge group G of horocycle scalings acts on the decorated Teichmüller space. Namely, we can multiply the height of the horocycle about point P by a positive factor $\lambda(P)$. We thus get an action of the multiplicative group $G = (\mathbb{R}_+)^s$ on the decorated Teichmüller space, which is described in Penner coordinates as follows:

(3.3) $$(\lambda_{P_1}, \ldots, \lambda_{P_s}) : f(e) \mapsto \lambda_{P_i} \cdot \lambda_{P_j} \cdot f(e),$$

where $e = P_i P_j \in \text{Edge}(\Delta)$, see [Pe1].

Consider an arbitrary cluster algebra $\mathcal{A}(Z)$. Assume that a n-tuple of integer weights $w_u = (w_{u,1}, \ldots, w_{u,n})$ is given at any vertex u of the tree $\mathbb{T}_n$. We define a *local toric action* on the cluster C_u at u as the $\mathbb{R}^\star$-action given by the formula $\{f_{u,1}, \ldots, f_{u,n}\} \mapsto \{f_{u,1} \cdot t^{w_{u,1}}, \ldots, f_{u,n} \cdot t^{w_{u,n}}\}$. We say that local toric actions are *compatible* (in $\mathcal{A}(Z)$) if for any two clusters C_a and C_b the following diagram is commutative:

$$
\begin{array}{ccc}
C_a & \longrightarrow & C_b \\
t^{w_a} \downarrow & & t^{w_b} \downarrow \\
C_a & \longrightarrow & C_b
\end{array}
$$

where the horizontal arrows are defined by cluster transformations (2.1). In this case, local toric actions together define a global toric action on $\mathcal{A}(Z)$. In other words, local toric actions extend to a global one if all relations (2.1) are homogeneous with respect to this action.

It is easy to see that local toric actions (3.3) are compatible in $\mathcal{A}(\Sigma)$, and hence they can be extended to a global toric action. Lemma 2.3 in [GSV] claims that the dimension of the group of all global toric actions is equal to the corank of the transformation matrix Z. Thus $\text{corank}\, Z(\Delta) \geqslant s$. Let us prove that $\text{corank}\, Z(\Delta) = s$.

Let Δ be a nice triangulation of a genus g surface Σ with s punctures. Let $V(\Delta)$ be the vector space generated by the basis $\{x_e \colon e \in \text{Edge}(\Delta)\}$, where $x_e = \log f(e)$, G be the gauge group acting on $V(\Delta)$ by (3.3). The form ω can be written, up to a constant, as $\sum_{b \frown a} dx_a \wedge dx_b$. For any subset S of $\text{Edge}(\Delta)$ we put $V(S) = \text{span}\{x_e \colon e \in S\} \subseteq V(\Delta)$. We call a subset $S \subseteq \text{Edge}(\Delta)$ *representative* if the gauge group G acts on $V(S)$ faithfully and S is minimal with respect to this property. We denote by $\Gamma(S)$ the spanning subgraph of Δ with the edge set S.

Theorem 3.3. (i) *Let Δ be a nice triangulation of Σ, S be any representative subset of $\text{Edge}(\Delta)$, R be the complement of S in $\text{Edge}(\Delta)$. If $R \neq \emptyset$, the restriction of the form ω onto $V(R)$ is a nondegenerate 2-form.*

(ii) *Each connected component of $\Gamma(S)$ contains exactly one cycle and possibly a number of trees attached to it. The length of this cycle is odd.*

Proof. (i) Let $V_0(\Delta)$ denote the vector space generated by the basis $\{y_P \colon P \in \text{Vert}(\Delta)\}$, and define a map $\partial^* \colon V_0(\Delta) \to V(\Delta)$ by $y_P \mapsto \sum_{e=PQ} x_e$. Evidently, ∂^* is an injection. We want to prove that the complex $V_0(\Delta) \xrightarrow{\partial^*} V(\Delta) \xrightarrow{\zeta^*} V(\Delta)$ is exact, where the second map is given by the matrix $Z^*(\Delta) = -Z(\Delta)$. Equivalently, it is enough to prove that the dual complex $V(\Delta) \xrightarrow{\zeta} V(\Delta) \xrightarrow{\partial} V_0(\Delta)$ is exact.

The map ∂ is given by $x_e \mapsto y_P + y_Q$ for $e = PQ$, so $\ker \partial$ is generated by the vectors $\sum_{e \in \text{Edge}(\Delta)} \alpha_e x_e$ such that $\sum_{e=PQ} \alpha_e = 0$ for any point $P \in \text{Vert}(\Delta)$. Consider the image of the map ζ given by $Z(\Delta)$. For any $e \in \text{Edge}(\Delta)$ define the *alternating quadrangle* $q(e) \in V(\Delta)$ by $q(e) = Z(\Delta)x_e$; evidently, the image of ζ is spanned by $\{q(e) \colon e \in \text{Edge}(\Delta)\}$.

Let $C = (e_1, \ldots, e_{2k})$ be an even length cycle in the graph of Δ with a specified first edge e_1. Define $\text{alt}(C) \in V(\Delta)$ to be $\sum (-1)^i x_{e_i}$. Clearly, $\partial(\text{alt}(C)) = 0$. We claim that both $\ker \partial$ and $\text{Im}\, \zeta$ are spanned by $\text{alt}(C)$, where C runs over all even length cycles in the graph of Δ.

Let us show first that any $\mathrm{alt}(C)$ is a linear combination of alternating quadrangles. We start from the following particular case. Let PQR and PMN be two triangles of Δ sharing the vertex P. Define $b_P(Q, R; M, N) \in V(\Delta)$ by $b_P(Q, R; M, N) = \mathrm{alt}(PQRPMN) = x_{PQ} + x_{PR} - x_{QR} - x_{PM} - x_{PN} + x_{MN}$. To prove that $b_P(Q, R; M, N)$ is a sum of alternating quadrangles, we proceed as follows. Order the edges incident to P cyclically in such a way that $e_1 = PQ$, $e_2 = PR$, $\dots$, $e_{l-1} = PM$, $e_l = PN$. It is easy to see that $b_P(Q, R; M, N) = \sum_{i=2}^{l-1} q(e_i)$.

For an arbitrary even length cycle $C = (e_1, \dots, e_{2k})$ we proceed as follows. Let $e_i = P_i P_{i+1}$; pick Q_i in such a way that $P_i P_{i+1} Q_i$ is a triangle of Δ. Then $2\,\mathrm{alt}(C) = \sum (-1)^i b_{P_i}(P_{i-1}, Q_{i-1}; P_{i+1}, Q_i)$, and we are done.

Further, we fix a vector $x = \sum_{e \in \mathrm{Edge}(\Delta)} \alpha_e x_e$ such that

$$(3.4) \qquad \sum_{e=PQ} \alpha_e = 0 \qquad \text{for any } P \in \mathrm{Vert}(\Delta),$$

and prove that it can be represented as a linear combination of $\mathrm{alt}(C)$ for even length cycles C. Consider the subgraph Γ of the graph of Δ containing only edges with $\alpha_e \neq 0$. Assume first that Γ contains a simple even length cycle C, that is, all the vertices of C are distinct. Define $\alpha = \min_{e \in C} |\alpha_e|$, then either $x + \alpha\,\mathrm{alt}(C)$ or $x - \alpha\,\mathrm{alt}(C)$ satisfies condition (3.4), while the corresponding graph has at least one edge less than Γ. If Γ does not contain a simple even length cycle, then Γ is a tree of edges and simple odd cycles, that is, any two simple odd cycles have at most one common vertex (see [We, Ex. 4.2.18]). By the faithfulness of the action, Γ itself is not a simple odd cycle. Moreover, condition (3.4) implies that Γ does not contain pendant vertices, hence it contains at least two simple odd cycles. Evidently, for any two simple odd cycles C_1 and C_2 in Γ there exist vertices $P_1 \in C_1$ and $P_2 \in C_2$ and a simple path π with the endpoints P_1 and P_2 that does not have common edges with C_1 and C_2 (in particular, it may occur that $P_1 = P_2$ and π is an empty path). We then define $\alpha_i = \min_{e \in C_i} |\alpha_e|$ for $i = 1, 2$, $\alpha_\pi = \frac{1}{2} \min_{e \in \pi} |\alpha_e|$, set $\alpha = \min\{\alpha_1, \alpha_2, \alpha_\pi\}$ and proceed as before with the non-simple even length cycle C consisting of C_1, C_2, and π, the latter traversed in both directions. As a result, we again obtain a vector satisfying (3.4) such that the corresponding graph has at least one edge less than Γ, and the statement follows by induction.

It remains to prove that $V(R)$ intersects $\ker \zeta^*$ trivially. Indeed, let $x \in \ker \zeta^*$. By above, this means that $x = \partial^*(\sum_{P \in \mathrm{Vert}(\Delta)} \lambda_P y_P)$. If, in addition, $x \in V(R)$, then $\{\lambda_P \colon P \in \mathrm{Vert}(\Delta)\}$ acts trivially on $V(S)$, which contradicts to the faithfulness of the action.

(ii) Follows immediately from the faithfulness of the action. $\square$

3.5. Denominators of transformation functions and intersection numbers.

In conclusion, we would like to prove the following result that gives a nice interpretation of denominators of transition functions in terms of intersection points on the surface Σ. Let us express all cluster variables in $\mathcal{A}(\Delta)$ as rational functions in the cluster variables of the initial cluster. Recall that all the clusters in $\mathcal{A}(\Delta)$ are related to nice triangulations of Σ, while cluster variables correspond to edges of this triangulation. Abusing notation, we denote an edge of a triangulation and the corresponding cluster variable by the same letter. For a cluster variable p and an initial cluster variable x we denote by $\delta_x(p)$ the exponent of x in the denominator of p. Besides, let a and b be two edges possibly belonging to different nice trian-

gulations; we denote by $[a, b]$ the number of their inner intersection points (that is, the ones distinct from their ends).

Theorem 3.4. $\delta_x(p) = [x, p].$

Proof. Consider an arbitrary edge p and two triangles bordering on p (see Fig. 6). Denote by q the edge obtained by flipping p. We want to prove the following two relations:

$$(3.5) \qquad [p, x] + [q, x] = \max\{[a, x] + [c, x], [b, x] + [d, x]\},$$

$$(3.6) \qquad \delta_x(p) + \delta_x(q) = \max\{\delta_x(a) + \delta_x(c), \delta_x(b) + \delta_x(d)\}.$$

To prove (3.5), we consider all the inner intersection points of x with the sides a, b, c, d. These points break x into consecutive segments, each either lying entirely inside the quadrangle $abcd$, or entirely outside this quadrangle. We distinguish three types of segments lying inside the quadrangle. The segments of the first type intersect two adjacent sides of the quadrangle and exactly one of the diagonals p and q (see Fig. 6). Thus, the contribution of any such segment to both sides of (3.5) equals 1. The segments of the second type intersect two opposite sides of the quadrangle and both diagonals. Moreover, all segments of the second type intersect the same pair of the opposite sides, otherwise x would have a self-intersection. This means that each segment of the second type contributes 2 to both sides of (3.5). Finally, the segments of the third type start at a vertex and intersect the opposite diagonal and one of the opposite sides of the quadrangle. They are treated in the same way as the segments of the second type, the only difference being that their contribution to both sides of (3.5) equals 1. To get (3.5) is enough to sum up the contributions of all the segments.
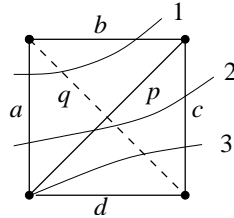

Fig. 6. Types of segments

To prove (3.6), consider an arbitrary cluster variable, say p, as a function of the initial cluster variable x, provided all the other initial cluster variables are fixed to 1. Then $p = P(x)/x^k$, where $P(x)$ is a polynomial in x with a nonzero constant term, and $k \geqslant 0$. It is easy to notice that the above constant term is positive. Indeed, by Theorem 3.1, $p > 0$ for any choice of positive values of the initial cluster coordinates. On the other hand, the sign of p coincides with the sign of the constant term under consideration, provided x is sufficiently small. Therefore, $\delta_x(ac + bd) = \max\{\delta_x a + \delta_x c, \delta_x b + \delta_x d\}$, and (3.6) follows immediately from this fact and the exchange relation $pq = ac + bd$.

Thus, we have proved that both $\delta_x(p)$ and $[p, x]$ satisfy the same relation with respect to flips. To get the statement of the theorem it suffices to notice that it holds evidently when p and e belong to two adjacent clusters. $\square$

This observation was independently made by Dylan Thurston, who used it, in particular, to describe relations between Penner coordinates corresponding to distant triangulations.

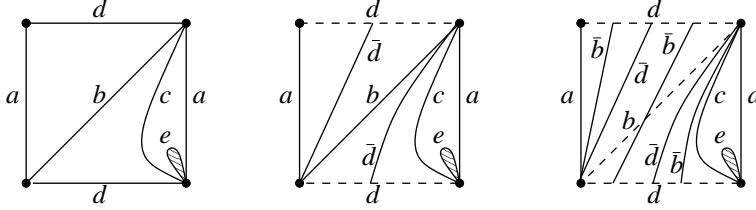We illustrate the lemma with the following example, see Fig. 7.

FIG. 7. DENOMINATORS AND THE NUMBER OF INTERSECTIONS

After the first move we have $\bar{d} = (ac + b^2)/d$, and hence $\delta_d(\bar{d}) = 1$, which corresponds to the intersection of the edges d and $\bar{d}$. After the second move we have $\bar{b} = (acd^2 + (ac + b^2)^2)/bd^2$ and hence $\delta_b(\bar{b}) = 1$, $\delta_d(\bar{b}) = 2$, which corresponds to the intersection of the edge $\bar{b}$ with b and to two intersections of $\bar{b}$ with d.

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